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Best Local Approximation

CHARLES K. CHUI

Department of Mathematics, Texas A&M University, College Station, Texas 77843

OVED SHISHA

Department of Mathematics, University of Rhode Island, Kingston, Rhode Island 02881

AND

PHILIP W. SMITH

Department of Mathematics, Texas A&M University, College Station, Texas 77843

1. INTRODUCTION

Let M be a class of functions in $C[0, \delta]$ with $\delta > 0$. All function spaces (classes) in this paper will be spaces of real-valued functions. Suppose that for each ϵ, $0 < \epsilon \leqslant \delta$, a function $f \in C[0, \delta]$ has a best uniform approximant $P_\epsilon(f)$ on $[0, \epsilon]$ from M; that is,

$$\| P_\epsilon(f) - f \|_{[0,\epsilon]} = \inf\{\| p - f \|_{[0,\epsilon]} : P \in M\},$$

where $\| \ \|_{[0,\epsilon]}$ is the supremum norm over $[0, \epsilon]$.

If, as $\epsilon \to 0^+$, $P_\epsilon(f)$ converges to some function $P_0(f)$ uniformly on some interval $[0, \epsilon_0]$, $\epsilon_0 > 0$, we say that $P_0(f)$ is a *best local approximant* of f.

In [1], the authors studied a situation where M is the class of all rational functions (with real coefficients) of type (m, n), $m \geqslant 0$, $n \geqslant 0$. It was shown that if $f \in C^{m+n+1}[0, \delta]$, $\delta > 0$, then the net $\{P_\epsilon(f)\}$, $0 < \epsilon \leqslant \delta$, converges, as $\epsilon \to 0^+$, to the $[m, n]$ Padé approximant of f, uniformly on some real neighborhood of 0.

In Section 2, we derive two basic properties of best local approximation when M is a finite-dimensional linear subspace of $C[0, \delta]$. In particular, we give a necessary and sufficient condition for the existence of best local approximants for all functions in $C^{m+n+1}[0, \delta]$. It turns out, as expected, that best local approximants from such a subspace display properties similar to those of Taylor polynomials.

In Section 3, we introduce the ideas of *best quasi-rational approximation* and *best local quasi-rational approximation.* As an application of a result in best quasi-rational approximation, we show that in the polynomial case the best local quasi-rational approximant of a "nonsingular" function in $C^{m+n+1}[0, \delta]$ coincides with its $[m, n]$ Padé approximant.

We think that the concepts of best local approximation and best local quasi-rational approximation are interesting. It is clear that many questions in this subject still remain unanswered.

The idea of "shrinking" intervals has been considered by many authors for different problems; we mention only Maehly and Witzgal [3, 4]. An interested reader also should refer to [5, Theorem 62 and Sects. 6.4, 9.3].

2. Best Local Approximation from a Haar Subspace

Throughout this section, we will let $u_1, ..., u_n$ be functions in $C^n[0, \delta]$, $\delta > 0$, such that $\{u_1, ..., u_n\}$ forms a Haar system on $[0, \delta]$. Let $S_n = S_n(u_1, ..., u_n)$ be the (algebraic) span of $\{u_1, ..., u_n\}$. For a function f in $C[0, \delta]$, $P_\epsilon(f)$, where $0 < \epsilon \leqslant \delta$, will denote the (unique) best uniform approximant of f on the interval $[0, \epsilon]$ from S_n; that is, $P_\epsilon(f)$ is the unique function P in S_n satisfying

$$\| f - P \|_{[0,\epsilon]} = \inf_{p \in S_n} \| f - p \|_{[0,\epsilon]} .$$

We have the following result concerning convergence of $P_\epsilon(f)$ to the best local approximant of f.

THEOREM 2.1. *The net $\{P_\epsilon(f)\}$, $0 < \epsilon \leqslant \delta$ converges as $\epsilon \to 0^+$ for every function $f \in C^n[0, \delta]$ if and only if the $n \times n$ matrix,*

$$A_n \equiv [u_j^{(i-1)}(0)] = \begin{bmatrix} u_1(0) & \cdots & u_n(0) \\ \vdots & & \vdots \\ u_1^{(n-1)}(0) & \cdots & u_n^{(n-1)}(0) \end{bmatrix}, \tag{2.1}$$

is nonsingular. Furthermore, in the case of convergence for a given $f \in C^n[0, \delta]$, the limit function $P_0(f)$ is in S_n and satisfies

$$P_0^{(j)}(f)(0) = f^{(j)}(0), \qquad j = 0, ..., n-1. \tag{2.2}$$

Observe that since S_n is finite-dimensional, coefficientwise convergence of $P_\epsilon(f)$ is equivalent to uniform convergence on $[0, \delta]$ and to pointwise convergence there. The matrix A_n is actually the Wronskian matrix of $u_1, \ldots, u_n$, evaluated at the origin. Hence, if A_n is nonsingular, then by continuity, the Wronskian matrix is nonsingular on some interval $[0, \eta]$, $\eta > 0$, so that $\{u_1, \ldots, u_n\}$ (or a rearrangement of it) is an extended Chebyshev system on $[0, \xi]$ for some $\xi > 0$ [cf. 2]. To establish Theorem 2.1, we need the following technical result.

LEMMA 2.1. *Suppose that the matrix A_n in (2.1) is nonsingular and that*

$$\left\| \sum_{i=1}^{n} \alpha_{i,\epsilon} u_i \right\|_{[0,\epsilon]} = o(\epsilon^{n-1}) \tag{2.3}$$

as $\epsilon \to 0^+$. Then, for each $i = 1, \ldots, n$, $\alpha_{i,\epsilon} \to 0$ as $\epsilon \to 0^+$.

Proof of Lemma 2.1. Without loss of generality, we may assume that for $i, j = 1, 2, \ldots, n$, we have $u_i^{(j-1)}(0) = \delta_{i,j}$, the Kronecker delta. Now suppose that

$$\gamma_\epsilon \equiv \max_{1 \leqslant i \leqslant n} |\alpha_{i,\epsilon}|$$

does not converge to zero as $\epsilon \to 0^+$. Let $\{\epsilon_k\}_{k=1}^{\infty}$ be such that $\epsilon_k \to 0$, and $0 < \epsilon_k \leqslant \delta$, $\gamma_{\epsilon_k} \geqslant \gamma > 0$ for all k. Let $\beta_{i,\epsilon_k} = \alpha_{i,\epsilon_k}/\gamma_{\epsilon_k}$. Then $|\beta_{i,\epsilon_k}| \leqslant 1$ for all i and k, and for each k, $|\beta_{i,\epsilon_k}| = 1$ for some $i = i(k)$. Clearly, for $i = 1, \ldots, n$,

$$u_i(t) = \frac{1}{(i-1)!} t^{i-1} + O(t^n)$$

as $t \to 0^+$. Hence,

$$\left\| \sum_{i=1}^{n} \beta_{i,\epsilon_k} \left(\frac{1}{(i-1)!}\right) t^{i-1} \right\|_{[0,\epsilon_k]} = \left\| \sum_{i=1}^{n} \beta_{i,\epsilon_k} u_i \right\|_{[0,\epsilon_k]} + O({\epsilon_k}^n).$$

On the other hand, since $\beta_{i,\epsilon_k} = \alpha_{i,\epsilon_k}/\gamma_{\epsilon_k}$ and $\gamma_{\epsilon_k} \geqslant \gamma > 0$, we have, from (2.3),

$$\left\| \sum_{i=1}^{n} \beta_{i,\epsilon_k} u_i \right\|_{[0,\epsilon_k]} = o(\epsilon_k^{n-1}).$$

Therefore, we can conclude that

$$\left\| \sum_{i=1}^{n} \beta_{i,\epsilon_k} \left(\frac{1}{(i-1)!}\right) t^{i-1} \right\|_{[0,\epsilon_k]} = o(\epsilon_k^{n-1})$$

with some β_{i,ϵ_k}, $i = i(k)$, having absolute value 1. Let $\{k_j\}_{j=1}^{\infty}$ be a sequence of $\{1, 2, \ldots\}$ and let i_0 be an integer, $1 \leqslant i_0 \leqslant n$, such that each $i(k_j) = i_0$.

Applying Markov's inequality i_0 times to the polynomials

$$\sum_{i=1}^{n} \beta_{i,\epsilon} \left(\frac{1}{(i-1)!}\right) t^{i-1},$$

we have $1 = |\beta_{i_0,\epsilon_{k_j}}| = o(1)$. Thus, we have proved that $\alpha_{i,\epsilon} \to 0$, as $\epsilon \to 0^+$, for $i = 1,..., n$.

Now we can prove Theorem 2.1. Suppose that $P_\epsilon(f) \to P_0(f)$ as $\epsilon \to 0^+$, for each $f \in C^n[0, \delta]$. Clearly, $P_0(f) \in S_n$. Let $0 < \epsilon \leqslant \delta$. Then, by the Alternation Theorem, there exist $\xi_{1,\epsilon},..., \xi_{n,\epsilon}$ such that $0 < \xi_{1,\epsilon} < \cdots < \xi_{n,\epsilon} < \epsilon$ and

$$P_\epsilon(f)(\xi_{i,\epsilon}) = f(\xi_{i,\epsilon}), \qquad i = 1,..., n.$$

Hence, by Rolle's theorem,

$$P_\epsilon(f)^{(j-1)}(\eta_{j,\epsilon}) = f^{(j-1)}(\eta_{j,\epsilon}), \qquad j = 1,..., n$$

where $0 < \eta_{1,\epsilon} < \cdots < \eta_{n-j+1,\epsilon} < \epsilon$. Therefore, we can conclude that

$$P_\epsilon(f)^{(j-1)}(0) \to f^{(j-1)}(0), \qquad j = 1,..., n,$$

as $\epsilon \to 0^+$. Since this holds for every $f \in C^n[0, \delta]$, we can choose arbitrary values of $f^{(j-1)}(0)$; that is,

$$\sum_{i=1}^{n} a_i u_i^{(j-1)}(0)$$

can assume any value for suitable $a_1,..., a_n$. Hence, the matrix A_n must be nonsingular.

Conversely, suppose that A_n is nonsingular and $f \in C^n[0, \delta]$. Write

$$P_\epsilon(f) = \sum_{i=1}^{n} \gamma_{i,\epsilon} u_i .$$

Again we may assume, without loss of generality, that $u_i^{(j-1)}(0) = \delta_{i,j}$; $i, j = 1,..., n$. By the definition of $P_\epsilon(f)$, we have

$$\begin{aligned} \| P_\epsilon(f) - f \|_{[0,\epsilon]} &\leqslant \left\| \sum_{i=1}^{n} f^{(i-1)}(0)\, u_i - f \right\|_{[0,\epsilon]} \\ &\leqslant \left\| \sum_{i=1}^{n} f^{(i-1)}(0) \frac{t^{i-1}}{(i-1)!} - f \right\|_{[0,\epsilon]} + O(\epsilon^n) \\ &= O(\epsilon^n). \end{aligned}$$

Hence, letting $\alpha_{i,\epsilon} = \gamma_{i,\epsilon} - f^{(i-1)}(0)$, we have

$$\left\| \sum_{i=1}^{n} \alpha_{i,\epsilon} u_i \right\|_{[0,\epsilon]} = \left\| P_\epsilon(f) - \sum_{i=1}^{n} f^{(i-1)}(0)\, u_i \right\|_{[0,\epsilon]}$$

$$\leqslant \| P_\epsilon(f) - f \|_{[0,\epsilon]} + \left\| \sum_{i=1}^{n} f^{(i-1)}(0)\, u_i - f \right\|_{[0,\epsilon]}$$

$$= O(\epsilon^n) = o(\epsilon^{n-1}).$$

By the above lemma, $\alpha_{i,\epsilon} \to 0$ as $\epsilon \to 0^+$, $i = 1,..., n$. Hence,

$$P_\epsilon(f) \to \sum_{i=1}^{n} f^{(i-1)}(0)\, u_i = P_0(f).$$

The last statement in the theorem follows by applying Rolle's theorem. This completes the proof of Theorem 2.1.

If the matrix A_n is singular, it is interesting to know what functions $f \in C^n[0, \delta]$ have the property that the net of best approximants $\{P_\epsilon(f)\}$ converges as $\epsilon \to 0^+$. To study this problem, we introduce the notion of *Taylor rank*. Assume that for some $N \geqslant n$,

$$U_N = \begin{bmatrix} u_1(0) & u_2(0) & \cdots & u_n(0) \\ u_1'(0) & u_2'(0) & \cdots & u_n'(0) \\ u_1^{(N-1)}(0) & u_2^{(N-1)}(0) & \cdots & u_n^{(N-1)}(0) \end{bmatrix}$$

exists and has rank n. Then the smallest such N is called the Taylor rank of the system $\{u_1 ,..., u_n\}$. Hence, if the matrix A_n in (2.1) is invertible, then $\{u_1 ,..., u_n\}$ has Taylor rank n.

We assume that for this smallest N, $u_1 ,..., u_n \in C^N[0, \delta]$. In general, let Ω_N be the image of R^n, the real Euclidean n-space, under the transformation U_N. We have the following result.

THEOREM 2.2. *Let $\{u_1 ,..., u_n\}$, have Taylor rank N, and let $f \in C^N[0, \delta]$ be such that the N-vector $\hat{f} = (f(0), f'(0),..., f^{(N-1)}(0))$ lies in Ω_N. Then the net $\{P_\epsilon(f)\}$ converges to $P_0(f) \in S_n$, as $\epsilon \to 0^+$, where $P_0(f)^{(j)}(0) = f^{(j)}(0)$ for $j = 0,..., N - 1$.*

When A_n of (2.1) is nonsingular, it is clear that $\hat{f} \in \Omega_N = R^n$ for every $f \in C^n[0, \delta]$. We also remark that Theorem 2.2 and its proof remain intact if $\{u_1 ,..., u_n\}$ is not a Haar system on $[0, \delta]$. Then $P_\epsilon(f)$ is *any* of the best uniform approximants of f on $[0, \epsilon]$ from S_n.

Proof of Theorem 2.2. Since $f \in \Omega_N$, we can find a function $P_0 \in S_n$ such that $P_0^{(j)}(0) = f^{(j)}(0)$ for $j = 0,..., N-1$. By Taylor's formula,

$$\| P_0 - f \|_{[0,\epsilon]} = O(\epsilon^N),$$

and hence, by the definition of $P_\epsilon(f)$,

$$\begin{aligned} \| P_0 - P_\epsilon(f) \|_{[0,\epsilon]} &\leqslant \| P_0 - f \|_{[0,\epsilon]} + \| P_\epsilon(f) - f \|_{[0,\epsilon]} \\ &\leqslant 2 \| P_0 - f \|_{[0,\epsilon]} = O(\epsilon^N). \end{aligned}$$

Write

$$P_0 - P_\epsilon(f) = \sum_{i=0}^{N-1} c_{i,\epsilon} t^i + O(t^N).$$

Then,

$$\left\| \sum_{i=0}^{N-1} c_{i,\epsilon} t^i \right\|_{[0,\epsilon]} = O(\epsilon^N).$$

Hence, by Markov's inequality,

$$c_{i,\epsilon} = O(\epsilon^{N-i}), \qquad i = 0,..., N-1,$$

so that $c_{i,\epsilon} \to 0$ as $\epsilon \to 0^+$, for $i = 0,..., N-1$. Let U_N^+ be a generalized inverse of U_N. Then

$$(P_0 - P_\epsilon(f))(t) = \sum_{j=1}^{n} a_{j,\epsilon} u_j(t)$$

where

$$(a_{1,\epsilon}, ..., a_{n,\epsilon}) = (c_{0,\epsilon}, ..., c_{N-1,\epsilon})[U_N^+]^{\mathrm{T}}$$

(the superscript T denotes, as usual, transpose). We remark that because of the Taylor rank N, any generalized inverse of U_N produces the same result. Now $a_{i,\epsilon} \to 0$ as $\epsilon \to 0^+$, for $i = 1,..., n$; so $P_\epsilon(f) \to P_0$ as $\epsilon \to 0^+$, and we have completed the proof of the theorem.

3. Quasi-Rational Approximation

Let P and Q be finite-dimensional subspaces of $C[0, \delta]$, $\delta > 0$, of dimensions m and n, respectively. We set

$$Q_0 = \{u \in Q : u(0) = 0\},$$

and assume throughout this section that

$$n - 1 = \dim Q_0 .$$

Set

$$Q_1 = \{u \in Q: u(0) = 1\}.$$

Then Q_1 is an affine variety in Q of dimension $n - 1$. Given $f \in C[0, \delta]$, consider the problem of minimizing

$$\| fq - p \|_{[0,\epsilon]} ; \qquad p \in P, \quad q \in Q_1 , \tag{3.1}$$

where $0 < \epsilon \leqslant \delta$. The minimizing pairs $(p_\epsilon , q_\epsilon)$, which exist as will be seen below, will be called best (uniform) quasi-rational approximants of f on $[0, \epsilon]$ from $P \times Q_1$, and the problem will be called best (uniform) quasi-rational approximation. We now state two results concerning existence and uniqueness of best quasi-rational approximants. They are almost self-evident.

PROPOSITION 3.1. *Let $f \in C[0, \epsilon]$. Then there exists a pair $(p_\epsilon , q_\epsilon) \in P \times Q_1$ such that*

$$\| fq_\epsilon - p_\epsilon \|_{[0,\epsilon]} = \inf\{\| fq - p \|_{[0,\epsilon]} : p \in P, q \in Q_1\}.$$

Choose $q_* \in Q_1$; then $Q_1 = q_* + Q_0$ and we see that the existence of a minimizing pair for (3.1) is equivalent to the existence of a minimum of

$$\{\| fq_* - u \|_{[0,\epsilon]} : u \in R_f\}, \tag{3.2}$$

where

$$R_f \equiv P + fQ_0 \equiv \{p + fq: p \in P, q \in Q_0\}, \tag{3.3}$$

a finite-dimensional subspace of $C[0, \delta]$. Clearly, (3.2) is minimized by some $u_\epsilon \in R_f$, so that (3.1) has a minimizing pair.

PROPOSITION 3.2. *Let $f \in C[0, \epsilon]$ be such that R_f as defined in* (3.3) *is a Haar subspace of* $C[0, \epsilon]$. *Then the problem of minimizing* (3.1) *has a unique solution.*

In order to consider best local quasi-rational approximation, (i.e., taking $\epsilon \to 0^+$), we must assume smoothness of the functions. Let $P, Q \subset C^{m+n+1}[0, \delta]$, $m \geqslant 1$, $n > 1$. Let $\{p_1 ,..., p_m\}$ and $\{q_1 ,..., q_{n-1}\}$ be bases of P and Q_0, respectively.

For every $f \in C^{m+n+1}[0, \delta]$, we let

$$\begin{aligned} \phi_i = \phi_i(f) &= P_i, \qquad \text{if} \quad 1 \leqslant i \leqslant m, \\ &= fq_{i-m}, \qquad \text{if} \quad m+1 \leqslant i \leqslant m+n-1. \end{aligned}$$

Then $\{\phi_1, ..., \phi_{m+n-1}\}$ spans R_f. Consider the Wronskian determinant of $\{\phi_i\}$, $i = 1, ..., m+n-1$, at the origin:

$$W(f, 0) \equiv W(f; \phi_1, ..., \phi_{m+n-1})(0) = \det[\phi_j^{(i-1)}(0)]. \tag{3.4}$$

If $W(f; \phi_1, ..., \phi_{m+n-1})(0) \neq 0$, then by continuity, we have $W(f; ..., \phi_{m+n-1})(x) \neq 0$ for all small $x \geqslant 0$, so that $\{\phi_1, ..., \phi_{m+n-1}\}$ (or a rearrangement of it), is an extended Chebyshev system [cf. 2]) on $[0, \epsilon]$ for some $\epsilon > 0$ and hence, is Haar there. Thus, if $W(f, 0) \equiv W(f; \phi_1, ..., \phi_{m+n-1})(0) \neq 0$, then the problem of minimizing (3.1) has a *unique* solution $(p_\epsilon, q_\epsilon) \in P \times Q_1$ for all sufficiently small $\epsilon > 0$. The following result will follow from Theorem 2.1.

THEOREM 3.1. *Let $f \in C^{m+n+1}[0, \delta]$ be such that $W(f, 0) \neq 0$. For all positive $\epsilon \leqslant$ some $\epsilon_0 \leqslant \delta$, let (p_ϵ, q_ϵ) be the (unique) best quasi-rational approximant of f on $[0, \epsilon]$ from $P \times Q_1$. Then the net $\{(p_\epsilon, q_\epsilon)\}$, $0 < \epsilon \leqslant \epsilon_0$, converges, as $\epsilon \to 0^+$, to a pair $(p_0, q_0) \in P \times Q_1$. Furthermore,*

$$(f - (p_0/q_0))(x) = O(x^{m+n-1}) \tag{3.5}$$

as $x \to 0^+$.

Proof. Since $W(f, 0) \neq 0$, $\{\phi_1, ..., \phi_{m+n-1}\}$ is a basis of R_f. Pick an arbitrary function $q^* \in Q_1$. By Theorem 2.1, the best uniform approximant u_ϵ of fq^* on $[0, \epsilon]$ converges as $\epsilon \to 0^+$. Now, $u_\epsilon = \hat{p}_\epsilon + f\hat{q}_\epsilon$, $\hat{p}_\epsilon \in P$ and $\hat{q}_\epsilon \in Q_0$. Note that $fq^* - u_\epsilon = f(q^* - \hat{q}_\epsilon) - \hat{p}_\epsilon$, and $q^* - \hat{q}_\epsilon \in Q_1$. By uniqueness, $q^* - \hat{q}_\epsilon = q_\epsilon$ and $\hat{p}_\epsilon = p_\epsilon$. Hence, the net (p_ϵ, q_ϵ) converges to, say, $(p_0, q_0) \in P \times Q_1$, as $\epsilon \to 0^+$. Furthermore, again from Theorem 2.1, we have

$$(p_0 + f(q^* - q_0))^{(j)}(0) = (fq^*)^{j}(0), \qquad j = 0, ..., m+n-2,$$

so that $(fq_0 - p_0)^{(j)}(0) = 0$, $j = 0, ..., m+n-2$. Since $q_0(0) = 1$, we can conclude that (3.5) holds as $x \to 0^+$.

As can be seen from the definition, there is a close connection between best local quasi-rational approximation and Padé approximation. Recall that if $f \in C^{m+n+1}[0, \delta]$, $\delta > 0$, $m \geqslant 0$, $n \geqslant 0$, then the $[m, n]$ Padé approximant of f, at 0, to be denoted by $[m, n](f)$, is the unique rational function $r = p/q$, where p and q are, respectively, polynomials of degrees $\leqslant m$ and $\leqslant n$, such that

$$(fq - p)^{(j)}(0) = 0$$

for $j = 0,..., m + n$. (Cf. [1, Lemma 1].) As an application of Theorem 3.1, we have the following result, relating best quasi-rational approximation and Padé approximation. To do this, consider $P = \text{span}\{1,..., x^m\}$ and $Q = \text{span}\{1,..., x^n\}$, so that $\dim P = m + 1$ and $\dim Q = n + 1$.

THEOREM 3.2. *Let $f(x) = a_0 + a_1x + \cdots + a_{m+n}x^{m+n} + O(x^{m+n+1})$ be a function in $C^{m+n+1}[0, \delta]$, $\delta > 0$, $m \geqslant 0$, $n \geqslant 0$, such that*

$$\det \begin{bmatrix} a_m & a_{m-1} & \cdots & a_{m-n+1} \\ a_{m+1} & a_m & \cdots & a_{m-n+2} \\ & \cdots & & \\ a_{m+n-1} & a_{m+n-2} & \cdots & a_m \end{bmatrix} \neq 0 \quad \text{and} \quad a_0 \neq 0. \qquad (3.6)$$

Here, $a_j = 0$ if $j < 0$. Let (p_ϵ, q_ϵ), $0 < \epsilon \leqslant \delta$, be the best quasi-rational approximant of f on $[0, \epsilon]$ from $P \times Q_1$ (which exists and is unique). Then the net $\{p_\epsilon/q_\epsilon\}$, $0 < \epsilon \leqslant \delta$, of rational functions of type (m, n) converges, as $\epsilon \to 0^+$, uniformly on some real neighborhood of 0, to the $[m, n]$ Padé approximant $[m, n](f)$ of f.

We remark that the nonsingularity condition (3.6) on f is a very common hypothesis in the literature on Padé approximation (cf, e.g., [6]).

To prove Theorem 3.2, we simply apply Theorem 3.1. Note that in this (polynomial) case, $\dim P = m + 1$ and $\dim Q = n + 1$ (instead of m and n, respectively, as in Theorem 3.1); $\dim Q_0$ is now $(n + 1) - 1 = n$. If we can prove that $W(f, 0) \neq 0$, then we shall have that (p_ϵ, q_ϵ) converges uniformly on some real neighborhood of 0, to (p_0, q_0), and by (3.5), we shall obtain

$$(f - (p_0/q_0))(x) = O(x^{m+n+1}),$$

so that, since $q_0(0) = 1$,

$$(fq_0 - p_0)^{(j)}(0) = 0, \qquad j = 0, 1,..., m + n,$$

that is, $p_0/q_0 \equiv [m, n](f)$. Recall that here

$$W(f) \equiv W(1,..., x^m, f \cdot x, f \cdot x^2,..., f \cdot x^n);$$

hence, as one easily can see,

$$W(f, 0) = \det \left[\begin{array}{ccc|cccc} 0! & & 0 & & & & \\ & \ddots & & & & \mathrm{X} & \\ 0 & & m! & & & & \\ \hline & & & (m+1)!\, a_m & (m+1)!\, a_{m-1} & \cdots & (m+1)!\, a_{m-n+1} \\ & \mathrm{O} & & (m+2)!\, a_{m+1} & (m+2)!\, a_m & \cdots & (m+2)!\, a_{m-n+2} \\ & & & & \cdots & & \\ & & & (m+n)!\, a_{m+n-1} & (m+n)!\, a_{m+n-2} & \cdots & (m+n)!\, a_m \end{array}\right]$$

$$= \prod_{j=0}^{m} (j!) \prod_{i=1}^{n} (m+i)! \det \begin{bmatrix} a_m & a_{m-1} & \cdots & a_{m-n+1} \\ a_{m+1} & a_m & \cdots & a_{m-n+2} \\ & \cdots & & \\ a_{m+n-1} & a_{m+n-2} & \cdots & a_m \end{bmatrix}$$

$$\neq 0$$

by (3.6). This completes the proof of the theorem.

4. Final Remarks

In Theorem 3.1, if $W(f, 0) \equiv W(f; \phi_1, ..., \phi_{m+n-1})(0) = 0$, then a best quasi-rational approximant (p_ϵ, q_ϵ) of f on $[0, \epsilon]$ from $P \times Q_1$ still exist for each ϵ, $0 < \epsilon \leqslant \delta$, but may not be unique, and in picking some best quasi-rational approximant (p_ϵ, q_ϵ) for every such ϵ, the net $\{(p_\epsilon, q_\epsilon)\}$ may or may not converge as $\epsilon \to 0^+$. Theorem 2.2 allows us to conclude that we do have convergence for functions f which are sufficiently smooth; for such an f and for a certain $q^* \in Q_1$, the vector

$$((q^*f)(0), (q^*f)'(0), ..., (q^*f)^{N-1}(0))$$

lies in the range of AR^{m+n-1}, where $A = [\phi^{(j)}(0)]$, $1 \leqslant i \leqslant m + n - 1$, $0 \leqslant j \leqslant N - 1$, and N is the Taylor rank of

$$\{\phi_1, ..., \phi_{m+n-1}\} \equiv \{p_1, ..., p_m, fq_1, ..., fq_{n-1}\}.$$

Some conclusions also can be drawn on the limit functions. However, many intriguing problems concerning best local approximation and best local quasi-rational approximation remain open.

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